Date: Tue, 04 Apr 2000 18:14:23 -0400 (EDT) From: Graham Solomon
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Subject: fulfillability
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Dear Allen, I had a bit of spare time ...

Notes taken by William Demopoulos from a lecture by Hilary Putnam, July 21 1983, on Kripke's concept of fulfillability.

Notation:

s = finite {<s(1), s(2), ..., s(n).} or infinite {<s(1), s(2), ..., s(n), ...>} of natural numbers.

s(i) is the i-th term of s

l(s) = n or oo, as appropriate, is the _length_ of s

{GS: I'll use A for the universal quantifier, E for the existential}

Illustration of our basic concept: s _fulfills_ the sentence phi

let phi = AxEyAz.psi(x,y,z) where psi is quantifier-free.

let s be a sequence of length v

Then we say s _fulfills_ phi if Ai<v [i.e., forall i lessthan v] the sentence phi' holds in N (= the standard model of PA) where
phi' = (Ax<s(i))(Ey<s(i+1))(Az<s(j).psi(x,y,z) where i is lessthanorequal j lessthan v

Note: were phi quantifier-free, s fulfills phi iff s satisfies phi (since phi = phi' in that case)

4 FACTS regarding _fulfills_:

FACT 1 phi is true iff there is an infinite increasing sequence s which fulfills phi.

(=>) if phi is true, _every_ sequence satisfies phi. Thus choose an increasing sequence with the appropriate properties

(<=) if s is infinite and increasing such that s fulfills phi then
(AxinN)(EyinN)(AzinN) N |= psi(x,y,z)
But then N |= phi

Definition: A sequence s is _good_ if

(i) s is finite [(En(l(s) = n)]

(ii) s(i)^2 lessthanorequal s(i+1) for i < l(s)

(iii) l(s) < s(1) [s is "large"]

Remark: (ii) ensures that s is closed under plus and times, in the sense that s fulfills (Ax,y)(Ez)(x+y=z) and ((Ax,y)(Ez)(x.y=z)

Definition: A sentence phi is _n-fulfilled_ if there is a good sequence s such that l(s) = n and s fulfills phi

Formally: (Es)(s is good & l(s) = n & s fulfills phi

Definition: phi is _fulfilled_ if (An)(phi is n-fulfilled)

FACT 2 phi is true ==> phi is fulfilled

If N |= phi then (An)(Es)(l(s) = n & ... & s fulfills phi). Let s be an infinite increasing sequence which fulfills phi (by FACT 1) such that s satisfies condition (ii) of good sequences. The problem is to find finite sequences which are good and which fulfill phi, one for each n. The key difficulty is to ensure that s(1) > l(s) for each finite s we choose. We proceed as follows:
if s(1) > n we let t = s|n+1 {where | is the half-arrow pointing up}
otherwise define t as follows:
t(1) = s(p)
.
.
.
t(k+1) = s(p+k)
where p is the first place where the value of s exceeds n. (We know that there must be such a p since s is infinite and increasing.)
Clearly t fulfills phi if s does.

FACT 3 FACT 2 is provable in PA. I.e., for each sentence phi PA|- phi -> (An)(Es)(l(s)=n & ... & s fulfills 'phi')

I.e., PA|- phi -> (An)(Es)(l(s)=n & ... &
(Ax<s(i))(Ey<s(i+1))(Az<s(j))psi(x,y,z)(Ai,j such that i lessthanorequal j lessthan n))

There is no difficulty with the quantification over finite sequences since such sequences can be coded by single numbers, thus eliminating the need for such quantification. So let s stand ambiguously for itself and for its Godel-number.

FACT 4 Facts 1-3 hold for the joint fulfillability of a finite number of sentences. I.e., phi is true and psi is true iff there is a sequence (infinite and increasing) which fulfills phi and which fulfills psi. If phi is true and psi is true then (An(Es)(phi is n-fulfilled & psi is n-fulfilled)

THEOREM: Let T be a theory such that
T is or contains PA
T + PA is w-consistent
T has an RE set of axioms {why RE, why not R?- because bounded quantifiers can define only up to RE?}

then 'T is fulfilled' is true in N but PA not|- 'T is fulfilled'

(thus 'T is fulfilled' is true in some other (non-standard) model of PA)

Cor. PA is not finitely axiomatizable

Proof of Cor. Let T finitely axiomatize PA
Since T finitely axiomatizes PA, PA|- phi_T, where phi_T is the conjunction of the axioms of T.
Hence: PA|- phi_T -> 'phi_T' is fulfilled, by FACT 3
PA|- phi_T
PA|- 'phi_T' is fulfilled
I.e., PA|- T is fulfilled, contradicting THEOREM

{next message will contain proof of THEOREM, the remaining half of these notes}